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COLLAPSE OF A ONE-DIMENSIONAL CAVITY*

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Isentropic potential flows arising when a one-dimensional cavity collapses into an ideally polytropic continuous medium are examined. The analysis is carried out up to the instant of focussing or up to the instant infinite gradient arise in the flow. As a result of the investigation of the analytic solutions in special variables for the polytropy exponent $1<\gamma<3$, it is proved that a free boundary separating the medium and the vacuum moves for some time with a constant velocity. Next, the solution is sought in physical space as a series in a neighborhood of the free boundary. When $\gamma>1$ it is proved that the series converges and the free boundary's acceleration begins only from the instant of origin of an infinite gradient. An ordinary differential equation is obtained, governing the behavior of the gradient on the free boundary. The solutions of this equation are studied by numerical calculations and particular solutions are found. It turned out that the instant of origin of the singularity depends in an essential manner on the initial data. Exponents γ_* are introduced, such that when $\gamma\leqslant\gamma_*$ there are no singularities on the free boundary up to the focussing instant if at the initial instant the medium was homogeneous and at rest. When $\gamma>\gamma_*$ the function $t_*=t_*(\gamma)$, viz., the instant of origin of an infinite gradient on the free boundary, is obtained. Calculations carried out by a difference scheme showed that when $\gamma\leqslant\gamma_*$ up to the focussing instant and when $\gamma>\gamma_*$ up to instant t_* there are no large gradient inside the whole flow region. The exponents γ_* coincide with those found earlier papers (**) in which the problem being investigated was investigated by means of constructing several terms of asymptotic series. It is concluded that when $1<\gamma<3$ the free boundary moves for some time $t_*>0$ at a constant velocity, and when $\gamma\leqslant\gamma_*=1+2/i$ the time t_* coincides with the focussing instant $t_1=(\gamma-1)/2$ and $t_*<t_1$ when $\gamma>\gamma_*$. The complete construction of the asymptotic expansions and the exact estimates of these expansions was not carried out. Power series solving the original problem in the exact statement are constructed recurrently in this paper. All the facts obtained are proved on the basis of the study of the convergence domains of these series. When $1<\gamma<3$ they coincide with earlier derivations.

1. We examine an at-rest homogeneous ideally polytropic medium with exponent $\gamma>1$ and a vacuum separated by a surface (a cylinder or a sphere, where the vacuum is inside and the medium is on the outside). At instant $t=0$ the surface is instantaneously removed and a one-dimensional flow of the medium into the vacuum commences. The region of the flow being examined, is bounded, on the one, hand, by the free boundary separating the medium from the vacuum and moves with the medium's velocity on this boundary, and, on the other, by a weak discontinuity which separates it from rest and moves with the velocity of sound c_0 in the medium at rest. In the planar case the solution is a centered simple rarefaction wave with linear profiles of the velocity and of the velocity of sound; the free boundary moves with constant velocity $2c_0/(\gamma-1)$ /1/.

It is difficult to solve the stated problem in the physical space (x, t) because of the presence at $t=0$ of an infinite gradient which is instantaneously "smeared", and the flow resembles a centered wave. Therefore, as in /2/, we can consider the boundary-value problem for the analog of the potential $\Psi(r, t)$ in the space of the variables (r, t) (r is the medium's velocity, $i=2$ corresponds to cylindrical symmetry, $i=3$, to sperical)

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**) KAZHDAN Ia.M., Spherical scattering of a gas toward the center. Moskow, Inst. Prikl. Mat. Akad. Nauk SSSR, Preprint No.2, 1969. ZHITNIKOV Iu.V. and KAZHDAN Ia.M., Convergence of a cylindrical rarefaction wave to the center. Moscow, Inst. Prikl. Mat. Akad. Nauk SSSR, Preprint No. 150.

$$\Psi_{tt}\Psi_{rr}\Psi_r + (i-1)(\gamma-1)\, r\Psi_{rr}(\Psi_t - r^2/2) = \Psi_r\,[(\Psi_{rt}-r)^2 - (\gamma-1)(\Psi_t - r^2/2)] \quad (1.1)$$
$$\Psi(0,t) = c_0^2 t/(\gamma-1), \quad \Psi_r(0,t) = c_0 t + R_0, \quad \Psi_{rr}(r,0) = 0$$

The conditions as $r=0$ are a consequence of the fact that a weak gap separates the flow from rest and moves with velocity c_0. The constant $R_0 > 0$ prescribes the surface's position at $t=0$. Without loss of generality we can take it that $c_0 = R_0 = 1$. The last condition in (1.1) is a consequence of the surface's instantaneous removal. In a neighborhood of point $(r=0, t=0)$ problem (1.1) has a unique analytic solution /3/, and the convergence domain of the series in powers of r is unbounded in t /4/. The flow in the physical space can be recovered by the formulas ($\Phi(x,t)$ is the flow's potential, c is the velocity of sound)

$$x = \Psi_r, \quad \Phi_x = r, \quad c^2 = (\gamma-1)(\Psi_t - r^2/2)$$

We construct the solution of problem (1.1) as

$$\Psi(r,t) = \sum_{k=0}^{\infty} \frac{T_k(r)\, t^k}{k!} \quad (1.2)$$

Then the initial and boundary conditions in (1.1) become the relations

$$T_0(0) = 0, \quad T_1(0) = 1/(\gamma-1), \quad T_k(0) = 0$$
$$T_0'(0) = 1, \quad T_1'(0) = 1, \quad T_k'(0) = 0, \quad k \geqslant 2; \quad T_0''(r) = 0$$

Therefore, $T_0(r) = r$. Having set $t=0$ in Eq.(1.1) and allowed for the form of $T_0(r)$, we obtain

$$(T_1' - r)^2 = (\gamma-1)(T_1 - r^2/2)$$

Using the condition for $T_1(0)$ (the condition on $T_1'(0)$ will be automatically fulfilled), we finally find

$$T_1(r) = \frac{1}{\gamma-1}\left(\frac{\gamma-1}{2} r \pm 1\right)^2 + \frac{r^2}{2}$$

Henceforth, we choose the plus sign corresponding to the collapse of the cavity (the minus sign corresponds to expansion). Having differentiated Eq.(1.1) with respect to t, set $t=0$ and allowed for the forms found for $T_0(r)$ and $T_1(r)$, we obtain for $T_2(r)$ the equation

$$T_2' - (3\gamma-1)T_2/(4v) = (i-1)(\gamma-1)rv/4$$
$$v = 1 + (\gamma-1)\, r/2$$

The condition $T_2(0) = 0$ ensures the solution's uniqueness, while the condition $T_2'(0) = 0$ is fulfilled automatically. As a result

$$T_2(r) = \frac{(i-1)(\gamma+1)}{(\gamma-1)^2}\left[\frac{v^{\alpha+1}}{(2-\alpha)(1-\alpha)} + \frac{v^3}{2-\alpha} - \frac{v^2}{1-\alpha}\right] \quad \text{when} \quad \gamma \neq \frac{5}{3}, 3; \quad \alpha = \frac{\gamma+1}{2(\gamma-1)}$$
$$T_2(r) = 6(i-1)v^2\left[v \ln v - \frac{r}{3}\right], \quad \gamma = \frac{5}{3}$$
$$T_2(r) = (i-1)v^2[r - \ln v], \quad \gamma = 3$$

When integrating, $|v|$ appears and to avoid unnecessary awkwardness we consider the case $r \geqslant -2/(\gamma-1)$. Note that $\alpha > 1$ when $\gamma < 3$.

Having differentiated Eq.(1.1) k times ($k \geqslant 2$) with respect to t, set $t=0$ and allowed for the forms of T_0 and T_1, we obtain for T_{k+1} the equation

$$T_{k+1}' - \frac{(\gamma-1)(k\alpha+1)}{2v} T_{k+1} = \frac{1}{2v} F_{k+1}$$
$$F_{k+1} = \sum_{p=2}^{k} NT_p'' T_{q+2} + \eta_1 \sum_{p=2}^{k-1} NqT_p'' T_{q+1} + \gamma_1 k \sum_{p=2}^{k-1} C_{k-1}^p T_p' T_{q+1} +$$
$$\sum_{p=2}^{k} NT_p'' \sum_{l=2}^{q} C_q^l T_l' T_{q-l+2} + \gamma_1 k(k-1)\eta_1 T_k + \frac{\eta_2}{\gamma_2} v^2 T_k'' +$$
$$\eta_2 \sum_{p=1}^{k-2} NT_{p+1} T_q'' + \gamma_1 \eta_2 k T_k - \sum_{p=1}^{k-2} NT_q' \sum_{l=1}^{p-1} C_p^l T_{l+1}' T_{p-l+1} -$$
$$\eta_1 k \sum_{p=1}^{k-2} C_{k-1}^p T_{p+1}' T_q' - 2v \sum_{p=1}^{k-2} NT_{p+1}' T_q' - 2v\eta_1 k T_k' +$$

$$\gamma_2 \sum_{p=1}^{k-2} NT_{p+1}T_q' + \gamma_2\eta_1 kT_k - \sum_{p=1}^{k-1} NT_{p+1}'T_{q+1}'$$
$$\gamma_1 = \frac{\gamma+1}{2}, \quad \gamma_2 = \gamma - 1, \quad q = k - p, \quad N = C_k^p$$
$$\eta_1 = \frac{\gamma+1}{\gamma-1} v - \frac{2}{\gamma-1}, \quad \eta_2 = 2(i-1)(v-1)$$

If in F_{k+1} the upper limit in the sums is less than the upper, then such a sum equals zero by definition. We note that T_l, T_l', T_l'' with $2 \leqslant l \leqslant k$ occur in F_{k+1}. The initial conditions $T_{k+1}(0) = 0$ $(k \geqslant 2)$ ensure the solutions' uniqueness, while the conditions $T_{k+1}'(0) = 0$ are automatically fulfilled. This can be shown by induction, taking into account that either r or T_l or T_l' $(2 \leqslant l \leqslant k)$ enter as multipliers in each summand of F_{k+1}. The form of $T_{k+1}(r)$ can be written out in terms of a quadrature

$$T_{k+1}(r) = v^{k\alpha+1}\left[C_{k+1} + \frac{1}{\gamma-1}\int F_{k+1}(v)\,v^{-k\alpha-2}\,dv\right]$$

where the C_{k+1} is uniquely found from the initial conditions. Series (1.2) solves problem (1.1); the series coefficients are locallay analytic functions of r. Therefore, series (1.2) is a locally convergent over-expansion in powers of t of the analytic solution of the problem being examined. For a more exact description of the convergence domain we consider the structure of $T_k(r)$.

Assertion. When $k \geqslant 1$ the coefficients of series (1.2) have the form

$$T_{k+1} = v^2 P_{k+1,1}(v, v^\alpha, v\ln v), \quad 1 < \gamma < 3$$
$$T_{k+1} = v^2 P_{k+1,2}(v, \ln v), \quad \gamma = 3$$
$$T_{k+1} = P_{k+1,3}(v, v^\alpha, v^{-1}, \ln v), \quad \gamma > 3$$

When $1 < \gamma < 3$ the leading power of v in T_{k+1} equals $\max\{k+2, \alpha k + 1\}$. Here $P_{n,m}(v, w, \ldots)$ is a polynomial in its variable, n, m denote the polynomial's number but not its degree.

The proof is carried out in several stages of induction. The inducation's base follows from the form of $T_2(r)$. At first is established that T_{k+1} is a polynomial of v, v^α, v^{-1}, $\ln v$ when $\gamma > 1$. This follows from the fact that for finding F_{k+1} we need to multiply out and add v, $v^{-k\alpha-2}$, T_l, T_l', T_l'' $(2 \leqslant l \leqslant k)$ which by the induction hypothesis have the needed form. When finding T_{k+1} the resultant expression is integrated, multiplied by $v^{k\alpha+1}$ and added to $C_{k+1}v^{k\alpha+1}$, i.e. the structure is preserved. The next step is to prove the presence of multipliers v^2 and the absence of negative powers of v when $\gamma \leqslant 3$. This is ensured by the fact that in F_{k+1} each summand has one of the following expressions as a multiplier: v^2, T_l, vT_l', $T_l' \times T_m'$ $(2 \leqslant l, m \leqslant k)$. After the multiplication of a summand of form $v^{2+\beta_1}\ln^{\beta_2} v$ $(\beta_1, \beta_2 \geqslant 0)$ from F_{k+1} by $v^{-k\alpha-2}$, integration and multiplication by $v^{k\alpha+1}$, the multiplier $v^{2+\beta_1}$ is preserved. In the summand $C_{k+1}v^{k\alpha+1}$ the power of v is automatically not less than two since $\alpha \geqslant 1$ when $\gamma \leqslant 3$ and $k \geqslant 2$. Finally, in case $\gamma < 3$ in T_l $(2 \leqslant l \leqslant k)$ we consider separately integer powers of v and powers of v in whose exponents α entered as a multiplier or a summand. In what follows these powers of v are called fractional powers of v. Of course, this division is somewhat conditional since there is a countable number of γ in the interval $1 < \gamma < 3$, for which α is an integer. Having assumed that in T_l $(2 \leqslant l \leqslant k)$ the maximum integer power of v equals $l+1$, we get that in F_{k+1} the maximum integer power equals $k+2$. After the multiplication by $v^{-k\alpha-2}$, v^{-1} can appear under the integral only at this maximum power of v (since $\alpha > 1$). But this is possible only when $\alpha = 1 + 1/k$, i.e., when $\gamma = 2 + (k-2)/(k+2)$. Hence, after the integration, $\ln v$ can appear only for these values of γ. If these values of γ have still not been examined, then after integration of these summands $\ln v$ does not appear and the maximum integer power of v in T_{k+1} is $k+2$.

Let us consider the fractional powers of v. Having assumed that in T_l $(2 \leqslant l \leqslant k)$ there are only the powers $(l-1)\alpha + 1 - l_1$, $(l-2-l_2)\alpha + (2+l_2) - l_3$, $l_1, l_2, l_3 \geqslant 0$, we get that in F_{k+1} there are only the powers $k\alpha - l_1$, $(k-1-l_2)\alpha + (2+l_2) - l_3$. After the multiplication of these powers of v by $v^{-k\alpha-2}$, v^{-1} does not appear since $\alpha > 1$. Consequently, after integration and multiplication by $v^{\alpha k+1}$ the powers of v are preserved in the summands being considered, and, in addition, the summand $C_{k+1}v^{k\alpha+1}$ is added to T_{k+1}. By the same token we have established that when $\gamma \neq 2 + (k-2)/(k+2)$, $k \geqslant 2$ $\ln v$ does not appear in T_{k+1}, and the maximum power of v equals $\max\{k+2, k\alpha+1\}$. Now let $\gamma = 2 + (k_0-2)/(k_0+2)$ with a fixed $k_0 \geqslant 1$. Then, once again examining integer powers of v and fractional powers of v separately, we get that $\ln v$ does not appear in T_l when

$l < k_0 + 1$, the highest integer power of v is $l+1$, and the highest fractional power of v is $\max\{l+1, (l-1)\alpha+1\}$. When $k = k_0$ the highest integer power of v in F_{k+1} is k_0+2 and after integration the summand $v^{k_0\alpha+1}\ln v$ appears in T_{k+1} . But $\alpha = 1+1/k_0$, therefore, the summand has the form $v^{k_0+2}\ln v$, i.e., the highest integer power is once again k_0+2. When integrating the fractional powers, $\ln v$ does not arise. When constructing the succeeding T_{k+1} $(k > k_0)$, $\ln v$ and its powers in F_{k+1} has as factors powers of v such that after multiplication by $v^{-\alpha k-2}$ summands of form $v^{-1}\ln^{\beta} v$ do not appear since $k > k_0$, $\alpha > 1$, i.e., $\ln v$ does not grow because of the integration, and, therefore, the powers of $\ln v$ grows no faster than the powers of v when $\gamma \neq 2 + (k-2)/(k+2)$. Having assumed that $T_l = v^2 P_l(v, v\ln v)$ and using the quadrature for T_{k+1}, we get that $T_{k+1} = v^2 P_{k+1}(v, v\ln v)$. The assertion has been proved.

Since series (1.2) converges locally, while when $1 < \gamma < 3$ the series coefficients are polynomials of v and $v\ln v$ and the degrees of the polynomials are not higher than $\max\{k+2, \alpha k+1\}$, it can be proved, as in /4/, that a constant $M > 0$ exists such that series (1.2) and the series for Ψ_r, Ψ_t, Ψ_{rr}, Ψ_{rt}, Ψ_{tt} converge in the domain

$$M\xi^{\alpha}t < 1, \quad \xi = \max\{1, v, |v\ln v|\}$$

In particular, the series listed yield a solution of problem (1.1) for $-2/(\gamma-1) \leqslant r \leqslant 0$, $0 \leqslant t \leqslant t_0$, $t_0 > 0$. The quantities T_{k+1} $(k \geqslant 1)$ have co-factor v^2, therefore, when $r = -2/(\gamma-1)$ we have $c = 0$, $x = 1-2t/(\gamma-1)$. Thus, in the space of variables (x, t) series (1.2) restores the flow in the whole of the region being examined from the weak discontinuity up to the free boundary which moves toward the lessening of x with constant velocity $-2/(\gamma-1)$ for $0 \leqslant t \leqslant t_0$. In the general case t_0 is less than the focussing time $t_1 = (\gamma-1)/2$ of the free boundary into the center of symmetry. For other γ the convergence domain is prescribed by the formulas

$$M_1\xi_1^{\alpha_1}t < 1, \quad \xi_1 = \max\{1, v, |\ln v|\} \quad \gamma = 3$$
$$M_2\xi_2^{\alpha_2}t < 1, \quad \xi_2 = \max\{1, v, v^{-1}, |\ln v|\} \quad \gamma > 3$$

M_1, M_2, α_1, α_2 are positive constants. Therefore, when $t > 0$, $v = 0$ the series automatically diverge (the summands $t^k\ln^{\alpha_1 k} v$ and $t^k v^{-\alpha_2 k}$ when $t > 0$, $v \to 0$ grow as k grows), and on the basis of an analysis of these series it is not possible to derive any mathematically precise facts concerning the constancy of the flow velocity or the acceleration of the free boundary. We note that formally when $\gamma = 3$ we have $c = 0$ and $x = 1-2t/(\gamma-1)$ when $r = 2/(\gamma-1)$ i.e., formally here the free boundary's motion takes place with a constant velocity.

2. We now consider the flow in the space of (x, t). If for the equation

$$\Phi_{tt} + 2\Phi_{tx}\Phi_x + (\Phi_x^2 - c^2)\Phi_{xx} - (i-1)c^2\Phi_x/x = 0 \tag{2.1}$$
$$c^2 = 1 - (\gamma-1)\Phi_t - (\gamma-1)\Phi_x^2/2.$$

describing the flow's potential, we specify at $t = t_0 > 0$ the analytic initial conditions

$$\Phi(x, t_0) = \varphi_1(x), \quad \Phi_t(x, t_0) = \varphi_2(x), \quad x \geqslant x_0 = 1-2t_0/(\gamma-1) > 0 \tag{2.2}$$

then by the Cauchy–Kovalevskaya theorem the resultant problem has a unique analytic solution. The prescribing of conditions (2.2) is equivalent to prescribing the velocity of sound and the medium's velocity at $t = t_0$. Later on we study the question on whether the free boundary in the solution of problem (2.1), (2.2) moves for some time with constant velocity if the initial data (2.2) satisfy the additional conditions

$$c^2(x_0, t_0) = \frac{\partial c^2}{\partial x}(x_0, t_0) = 0, \quad \Phi_x(x_0, t_0) = -2/(\gamma-1) \tag{2.3}$$

(In particular, they are satisfied by the centered rarefaction wave and by the solution obtained in Sect.1). We study as well the question on whether the initial data are compatible with the solution resulting from the decay of the discontinuity when $t = 0$.

To answer the questions posed we make the change of variables $z = x + 2t/(\gamma-1) - 1$, $t' = t$, i.e., as a new coordinate axis we take the assumed free boundary. In what follows we omit the prime. The conditions

$$\Phi_x(0, t) = -2/(\gamma-1), \quad c^2(0, t) = 0$$

are prescribed when $z = 0$. As a result we obtain the problem

$$\Phi_{zz}\left[\frac{4}{(\gamma-1)^2} + \frac{4}{\gamma-1}\Phi_z + \Phi_z^2 - c^2\right] + 2\Phi_{zt}\left(\frac{2}{\gamma-1} + \Phi_z\right) +$$
$$\Phi_{tt} - (i-1)c^2\Phi_z/[1 - 2t/(\gamma-1) + z] = 0$$
$$c^2 = 1 - 2\Phi_z - (\gamma-1)\Phi_t - (\gamma-1)\Phi_z^2/2$$
$$\Phi(0, t) = (\gamma+1)t/(\gamma-1)^2, \quad \Phi_z(0, t) = -2/(\gamma-1)$$

which is a characteristic Cauchy problem and, therefore, has a nonunique solution. Having represented the solution as

$$\Phi(z,t) = \sum_{k=0}^{\infty} a_k(t) z^k \tag{2.4}$$

we get that the coefficients $a_k(t)$ must satisfy the ordinary differential equations

$$\begin{gathered}(a' + a^2)' + (\gamma+1) a (a' + a^2) - \frac{2(i-1)(a'+a^2)}{1-2t/(\gamma-1)} = 0, \qquad a = 2a_2 \\ a''_{k+1} + a'_{k+2}\{(\gamma-1)a + 2(k+2)a - 2(i-1)/[1-2t/(\gamma-1)]\} + \\ (k+2)a_{k+2}\left\{(k+1)\left(\frac{\gamma-1}{2}a' + \frac{\gamma+1}{2}a^2\right) + (\gamma+1)a^2 + \right. \\ \left. 2a' - 2(i-1)a/[1-2t/(\gamma-1)]\right\} = \\ P_{k+2}\left(\frac{1}{1-2t/(\gamma-1)}, a_2 a_2', a_2'', \ldots, a_{k+1}, a'_{k+1}, a''_{k+1}\right), \quad k \geqslant 1\end{gathered} \tag{2.5}$$

Because it is unwieldy we do not present here the actual form of P_{k+2}. The equations for a_k $(k \geqslant 3)$ are linear and, therefore, the singularities of their solutions can hold only at the focussing instant when $t = t_1$ and further at those instants when there are singularities in $a_2(t)$. If for these equations the initial conditions

$$a_k(t_0) = C_{k0}, \quad a_k'(t_0) = C_{k1}, \quad k \geqslant 2$$

are prescribed, such that the series

$$\Phi_{zz}(z, t_0) = \sum_{k=0}^{\infty} (k+2)(k+1) C_{k+2,0} z^k$$

$$\Phi_{zzt}(z, t_0) = \sum_{k=0}^{\infty} (k+2)(k+1) C_{k+2,1} z^k$$

converge locally, then the problem posed for $\Phi(z, t)$ has /3/ a unique locally analytic solution. The prescribing of conditions on $a_k(t_0)$, $a_k'(t_0)$ $(k \geqslant 2)$ is equivalent to prescribing conditions (2.2) and (2.3). Then from the uniqueness of the analytic solution follows the coincidence of the solutions of problem (2.1)—(2.3) and of the problem for $\Phi(z, t)$. By the same token we have proved that the free boundary on problem (2.1)—(2.3) for any $\gamma > 1$ still moves with constant velocity for some time $t_0 \leqslant t \leqslant t_*$, $t_* > t_0$

If the discontinuity prescribed at instant $t = 0$ decomposes so as to form a locally analytic isentropic flow, then the right-hand sides of conditions (2.2) are uniquely determined by this flow. To find the instant t_* it is necessary to describe in detail the convergence domain of series (2.4). A detailed investigation of this question is not presented here because of its awkwardness. We merely point out that the boundary point t_* of the convergence domain of the series (as $t \to t_*$ the radius of convergence of the series tends to zero as some positive power of the difference $|t - t_*|$) coincides with the instant of origin of a singularity in the solution of Eq.(2.5) for $a(t)$. This fact is proved along the following scheme.

The solution of Eq.(2.5) can be majorized by a function of form

$$M / [1 - (t - t_0) / \rho]$$

where the constant $\rho > 0$ specifies the distance from t_0 to the closest singularity in $a(t)$ that can lie to the right or to the left of t_0. The solutions of the equations for $a_k(t)$ $(k \geqslant 3)$ can be majorized by polynomials of

$$v_1 = 1/[1 - 2t/(\gamma-1)], \quad v_2 = 1/[1-(t-t_0)/\rho], \quad \ln v_1, \quad \ln v_2$$

The degrees of these polynomials grow no faster than Ak, where the constant A is determined by the form of Eq.(2.1). Therefore, a boundary point of the convergence domain of the majorizing series, and, hence, of series (2.4), is the instant, closest to t_0, of the origin of singularities in the solution of the equation for $a(t)$. If such a closest instant is located, say, to the left of t_0, then by using the solution obtained we can set analytic initial conditions for $\Phi(z, t)$ at a new instant $t = t_0 + \rho - \varepsilon$, where ε is a positive quantity much less than ρ. Having done this the number of times needed, we get that the instant of origin of a singularity to the right of the initial instant lies closer to it than the one found to the left. Consequently, the boundary points of the convergence domain of the series are the instant of origin of singularities in $a(t)$ independently of which side and of what distance from the initial t_0 they lie.

Note. Several terms (no more than two or three) of various asymptotic approximations of similar problems have been constructed and investigated earlier (*). If we look upon these terms as the start of infinite series, then in the notation adopted in the present paper these series are of the form

$$\sum_{k=0}^{\infty} f_k(z/t)\,t^k, \quad \sum_{k=0}^{\infty} g_k(t)\,(z/t)^{\delta_1 k}, \quad \sum_{k=0}^{\infty} h_k(z, t^{\delta_2})\,z^k$$

δ_1, δ_2 are constants. In the first case and in the second with $\delta_1 = 1$, f_0, f_1, g_0, g_1, g_2 coincide with the corresponding coefficients of series (2.4) if the latter starts off in analogous powers. If we have managed to construct recurrently all the f_k, g_k, h_k, then using the procedure proposed in the present paper and in /4/, we can most likely prove the convergence of these series in a neighborhood of the point $(z = 0,\ t = t_0)$, $t_0 > 0$. However, as was obtained for series (2.4) and for series (1.2), when $\gamma \geqslant 3$ we need to expect that the point $t = 0$ will be a boundary point for the convergence domain. Therefore, when $0 \leqslant t \leqslant t_0$ the convergence domain does not completely cover the region of the flow from the free boundary to the weak discontinuity.

Equation (2.5) is the so-called transport equation since it describes the behavior of the unknown function's gradient on a characteristic which in the case at hand serves as the free boundary separating the medium from the vacuum. Analogous equations in other variables have been derived previously for the cases $i = 3$ and $i = 2$ separately. The singular points of these equations have been analyzed with the aid of f_0 and f_1. Particular solutions have been obtained: $\gamma = 5/3$ when $i = 3$ and $\gamma = 5/3, \gamma = 2$ when $i = 2$

3. Let us consider Eq. (2.5) for $a(t)$ and clarify the question on the instants at which singularities arise in the solution. We see that Eq. (2.5) admits of the special solutions

$$a = 0; \quad a = i\,[1 - (i+1)\,t]\,/\,[(i+1)\,t\,(1 - it)] \quad \text{when} \quad \gamma = \gamma_* = 1 + 2/i;$$

$$a = [2\,(i-1)(\gamma-1) - 4]\,/\,[(\gamma-1)^2\,(1 - 2t/(\gamma-1))];$$

$$a = 1/(t + C)$$

where C is an arbitrary constant. If when prescribing the initial conditions (2.2) the constants C_{20} and C_{21} are taken in accordance with the first three particular solutions found, then the focussing instant $t = t_1$ will be a boundary point for the convergence domain. If the constants indicated are chosen in accordance with the last particular solution, then the singularities on the free boundary will be at instants $t = t_1$ and $t = -C$. Since C is arbitrary, we can take $C = 0$. As the constants C_{20} and C_{21} we specify the values of $\partial^2 c^2/\partial x^2$ and $\partial^3\Phi/\partial x^2$ at the point (x_0, t_0). A countable collection of constants C_{k0} and C_{k1} $(k \geqslant 3)$ still remain ; with their aid we can give the global behavior of the profiles $\Phi_x(x, t_0)$ and $c^2(x, t_0)$: ensure the necessary monotonicity of the profiles, satisfy the conditions of continuity on the weak discontinuity, specify the mass of the medium moving at instant $t = t_0$, etc. Hence, under certain special initial data the free boundary moves with constant velocity for any $\gamma > 1$ (if, of course, singularities do not arise outside the convergence domain of the series). In addition, in the planar case $(i = 1)$ there are two particular solutions to Eq. (2.5) when $\gamma = 3$: $a = 1/(2t)$, $a = (1 - 2t)/[2t(1 - t)]$, with like asymptotic behavior as $t \to 0$. But these functions yield qualitatively different solutions of problem (2.1)—(2.3): the first is a rarefaction wave and the second is a flow in which an infinite gradient arises when $t = 1$ Thus, the instant of origin of a singularity in the solutions of Eq. (2.5) essentially depends on the initial data.

For a subsequent analysis of the equation for $a(t)$ we can in standard fashion exclude t from Eq. (2.5) and, next, lower the equation's order. We have hardly succeeded in computing the general solution of Eq. (2.5) /5/. With the replacement

$$y(t) = \exp\left(\int_{t_0}^{t} a(t)\,dt\right)$$

we obtain the problem

$$[1 - 2t/(\gamma-1)]^{(i-1)(\gamma-1)}\,y^{\gamma} y'' = C_1; \quad y(t_0) = 1, \quad y'(t_0) = C_2, \quad C_1 = a'(t_0) + a^2(t_0), \quad C_2 = a(t_0)$$

convenient for a numerical analysis. The initial values $a(t_0)$, $a'(t_0)$ must be chosen from the solution of the discontinuity decay problem which has been solved with mathematical rigor only

*) See the footnote on p.41.

for $1 < \gamma < 3$. The solution found when t is close to zero is only slightly different from the solution in the planar case; therefore, for all γ in the calculations we chose initial data corresponding to the centered wave /1/, i.e.,

$$C_1 = -2(\gamma - 1) / [(\gamma + 1) t_0]^2, \quad C_2 = 2 / [(\gamma + 1) t_0]$$

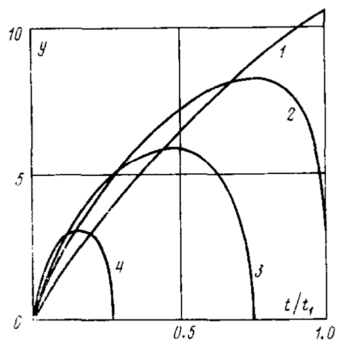

Fig.1

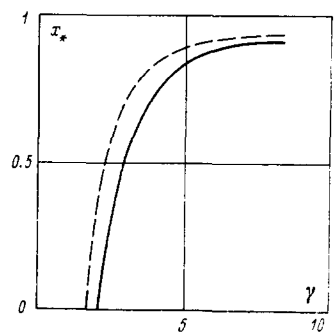

Fig.2

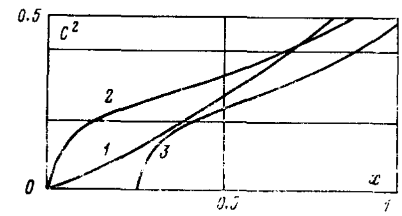

Fig.3

Fig.1 shows the graphs of the calculations of $y(t)$ for $i = 3$, $t_0 = 0{,}01$. Curves 1–4 correspond to the values of γ: 1.4; 1.7; 2; 3. For $\gamma \leqslant \gamma_0$ ($\gamma_0 \approx 1.72$ when $i = 2$; $\gamma_0 \approx 1.45$ when $i = 3$) the absolute value of the medium's velocity everywhere in the flow region is less than on the free boundary and the flow profiles are monotonic. When $\gamma > \gamma_0$ the absolute value of the medium's velocity immediately after the free boundary, beginning with the instant when $y'(t) = 0$, becomes larger than on the boundary, i.e., the medium next to the free boundary begins to accelerate and condense. When $t < t_0$ the singularity in the solution always arises at $t = 0$. For $\gamma \leqslant \gamma_* = 1 + 2/i$ the instant of origin of the singularity, to the right of t_0, coincides with the focussing instant t_1. These values of γ_* coincides with those found earlier (*). When $\gamma > \gamma_*$ an infinite gradient arises on the free boundary at the instant t_* $(0 < t_* < t_1)$. Fig.2 shows the function $x_*(\gamma) = 1 - t_* / t_1$ (the solid curve corresponds to $i = 2$ and the dashed, to $i = 3$). We recall that when $t = 0$ the free boundary was located at $x = 1$. Thus, when $\gamma \leqslant \gamma_*$ a boundary point for the convergence domain of the series is the point $x = 0$, while when $\gamma > \gamma_*$ the point $x = x_*$.

To describe the whole flow region we can use for $0 \leqslant t \leqslant t_0$ the series (1.2) for $1 < \gamma < 3$ or the centered rarefaction wave for $\gamma > 1$, for $t_0 \leqslant t \leqslant t_*$ we can use series (2.4) in a neighborhood of the free boundary and the series for function $\psi(r, t)$ in powers of r in a neighborhood of the weak discontinuty /4/. All these solutions are well matched with each other in the middle part of the flow region. When $\gamma \geqslant 3$ the solution in a neighborhood of the free boundary has been constructed under an assumption on the corresponding decay of the initial discontinuity. Qualitatively, this solution is consistent with those found earlier: for large γ infinite gradients arise sufficiently rapidly in the flow and the free boundary begins to accelerate. Of course, we can use the series under the condition that in the middle part of the flow region no singularities arise. It can be shown that there will not be a weak discontinuity inside the flow region; therefore, it is necessary to observe only the origin of the free gradients. Rather obvious calculation by difference methods over the whole flow region showed that when $\gamma < \gamma_*$ there are no large gradients in the flow. When $\gamma \geqslant \gamma_*$ they arise on the free boundary at instants compatible with t_*. Fig.3 shows the dependence of the square of the medium's velocity on x when $i = 3$. Curves 1, 2, 3 correspond to the values: $\gamma = 1.4$, $t = t_1$; $\gamma = 1.7$, $t = t_* \approx t_1$; $\gamma = 2$, $t = t_* < t_1$. When $\gamma > \gamma_*$ the flow's properties of being potential and isentropic are violated at the instant $t = t_*$. After this the free boundary begins to accelerate at the expense of the medium becoming denser. Therefore, for a further description of the motion it is necessary, in general, to use the complete system of equations of gasdynamics, possibly, to introduce shock waves or weak discontinuities, and to make use of the solution obtain in the present paper for prescribing the initial conditions at $t = t_*$. The question on whether the solution of this new problem is selfsimilar still remains open.

In this paper we have considered the case of cavity collapse, but the majority of the formulas and calculations carry over without the principal changes to the case of expansion

*) See the footnote on p.41.

(the vacuum is outside and the medium inside).

REFERENCES

1. ROZHDESTVENSKII B.L. and IANENKO N.N., Systems of Quasilinear Equations and Their Application to Gasdynamics. Moscow, NAUKA, 1968.
2. BAUTIN S.P., Approximate calculation method for one-dimensional gas flows induced by non-monotonic motion of a piston. PMM Vol.40, No.6, 1976.
3. BAUTIN S.P., Characteristic Cauchy problem for a quasilinear analytic system. Differents. Uravnen., Vol.12, No.11, 1976.
4. BAUTIN S.P., Investigation of the convergence domain of special series solving certain gas-dynamics problems. In: Numerical Methods in the Mechanics of a Continuous Medium. Vol.9, No.4. Novosibirsk, Vychisl. Tsentr Sibirsk. Otdel. Akad. Nauk SSSR, 1978.
5. KAMKE E., Differentialgleichungen: Lösungsmethoden und Lösungen. Band 1: Gewöhnliche Differentialgleichungen. New York, Chelsea Publ. Co., 1959.

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